I. Why study analysis?

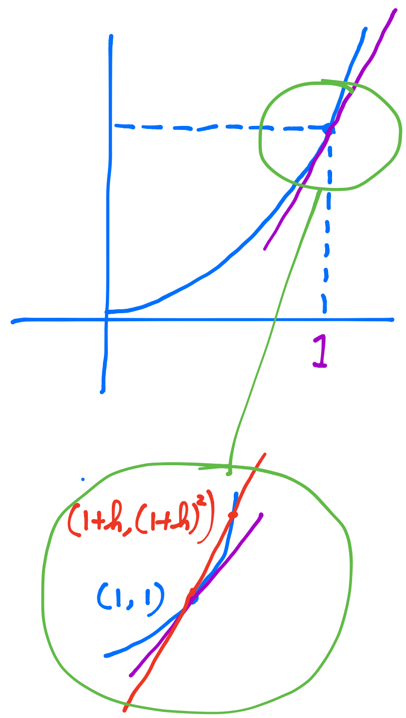
Newton (1660's) : Calculus w/o a solid foundation

What caused the problems? Infinitesimals.

<u>Example</u>: Given a parabola $y = x^2$.

Find the slope of the tangent line at $x = 1$.

Newton's argument:



Compute the slope of the secant line passing thru $(1, 1)$, $(1+h, (1+h)^2)$ where h is an "infinitesimal"

$$= \frac{(1+h)^2 - 1}{1+h-1} = \frac{h^2 + 2h}{h}$$

$= 2 + h$ (because h is nonzero).

$= 2$ (because h is infinitesimal)

What is an infinitesimal? Is it zero? Is it nonzero?

Berkeley (1730's): May we not call them the ghost of departed quantities?

Human-kind took almost 200 years to fix it.

At the end of the semester, you should also be able to fix it.

II. Numbers.

Ancient Greek wants to describe every length by numbers.

God created natural numbers (Kronecker)

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

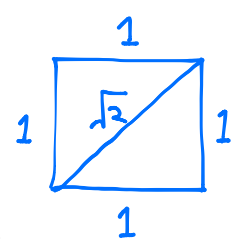
However it doesn't do the job: not closed under divisions.

How about rational numbers?

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \text{ integers, mutually coprime} \right\}$$

It doesn't do the job either!

Consider the diagonal of a square of side length 1.



Pythagorean Thm $\Rightarrow$ its length $= \sqrt{2}$.

Theorem: $\sqrt{2} \notin \mathbb{Q}$.

Proof by contradiction:

Assume $\sqrt{2} \in \mathbb{Q} \Rightarrow \sqrt{2} = \frac{m}{n}$, m, n coprime.

$\Rightarrow m^2 = 2n^2 \Rightarrow m^2$ is even $\overset{(*)}{\Rightarrow} m$ is even

$\Rightarrow \exists\, m_1$, s.t. $m = 2m_1 \Rightarrow 4m_1^2 = 2n^2$.

$\Rightarrow n^2 = 2m_1^2 \Rightarrow n^2$ is even $\overset{(*)}{\Rightarrow} n$ is even.

$\Rightarrow 2$ is a common factor of m, n, contradiction!

Key: If p is a prime number, $p \mid m^2$

then $p \mid m$.

(*) is the special case when $p = 2$.

In other words, $\mathbb{Q}$ is insufficient to describe all lengths.
The numbers that "really" work are called "real" numbers
denoted $\mathbb{R}$.

At this moment. $x \in \mathbb{R}$ corresponds to some length.

III. Preliminaries of the class:

(1) A set is any collection of distinct objects.

e.g. the set of people in this classroom. (25 elements)

the set of the schools people in this classroom attend = {School of Arts & Science, School of Engineering, Grad. School – Camden} (3 elements).

One can define a set by

(i) Listing all members, e.g., $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

(ii) Description in words, e.g., E = the set of even numbers.

(iii) Some kind of rules, e.g. $S = \{ r \in \mathbb{Q} : r^2 < 2 \}$

(2) Inclusion:

$$A \subseteq B \Leftrightarrow B \supseteq A \Leftrightarrow (\forall x \in A)\ x \in B.$$

A is referred as a <u>subset</u> of B.

Equation:

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$$

Empty set : a set with no elements.

(3) Unions

Given two sets A, B, the <u>union</u> $A \cup B$ is defined by

$$x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B).$$

Given a collection of sets $\{A_\lambda\}_{\lambda \in \Lambda}$, the <u>union</u> $\bigcup_{\lambda \in \Lambda} A_\lambda$ is defined by

$$x \in \bigcup_{\lambda \in \Lambda} A_\lambda \Leftrightarrow \exists \lambda \in \Lambda,\ x \in A_\lambda.$$

(4). Intersections is

Given two sets A, B, the <u>intersection</u> $A \cap B$ is defined by

$$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B).$$

Given a collection of sets $\{A_\lambda\}_{\lambda \in \Lambda}$, the <u>intersection</u>

$\bigcap_{\lambda\in\Lambda} A_\lambda$ is defined by

$$x \in \bigcap_{\lambda\in\Lambda} A_\lambda \Leftrightarrow \forall \lambda \in \Lambda,\ x \in A_\lambda.$$

Example: $A_1 = \mathbb{N} = \{1, 2, 3, \cdots\}$

$A_2 = \{2, 3, 4, \cdots\}$

$A_3 = \{3, 4, 5 \cdots\}$

Consist of inf. many elements.

$\vdots$

$A_n = \{n, n+1, n+2, \cdots\}$

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$$

Proof: $A_n \subseteq \mathbb{N}.\ \forall n. \Rightarrow \bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{N}.$

Recall: If $A \subseteq C$, $B \subseteq C$ then $A \cup B \subseteq C$.

Pf: $\forall x \in A\cup B \Rightarrow (x\in A) \vee (x \in B) \Rightarrow x \in C$.

Recall: If $\forall \lambda \in \Lambda$, $A_\lambda \subseteq B$ then $\bigcup_{\lambda\in\Lambda} A_\lambda \subseteq B$

Pf: $\forall x \in \bigcup_{\lambda\in\Lambda} A_\lambda \Rightarrow \exists \lambda,\ x \in A_\lambda \Rightarrow x \in B$.

$\forall x \in \mathbb{N},\quad x \in A_1 \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

Recall: $x \in A \Rightarrow x \in A\cup B$.

so $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$

Therefore $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.

$$\bigcap_{n=1}^{\infty} A_n = \phi.$$

<u>Proof by contradiction</u>: otherwise

$$\bigcap_{n=1}^{\infty} A_n = \phi \Rightarrow \exists x \in N, \quad x \in \bigcap_{n=1}^{\infty} A_n.$$
$$\Rightarrow \forall n \in N, \ x \in A_n$$

But $x \notin A_{x+1}$. contradiction.

<u>Example</u>: $A_0 = \{0\}$.

$A_1 = \{-1, 0, 1\}$.

$A_2 = \{-2, -1, 0, 1, 2\}$.

$\vdots \qquad \vdots$

$A_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$.

$\bigcap_{n=0}^{\infty} A_n = \{0\}$: ① $0 \in \bigcap_{n=0}^{\infty} A_n \Rightarrow \{0\} \subseteq \bigcap_{n=0}^{\infty} A_n$

② $\forall x \neq 0, \ x \notin \bigcap_{n=0}^{\infty} A_n. \Rightarrow \bigcap_{n=1}^{\infty} A_n \subseteq \{0\}$.

$\bigcup_{n=0}^{\infty} A_n = \mathbb{Z}$ ① $A_n \subseteq \mathbb{Z} \Rightarrow \bigcup_{n=0}^{\infty} A_n \subseteq \mathbb{Z}$

② $\forall x \in \mathbb{Z}, \ x \in A_x. \Rightarrow \bigcup_{n=0}^{\infty} A_n \supseteq \mathbb{Z}$.

(5) Complement:

Given $A \subseteq \mathbb{R}$, the <u>complement</u> of A, written A^c refers to the set of all elements of $\mathbb{R}$ NOT in A.

$$A^c = \{x \in \mathbb{R} : x \notin A\}.$$

De Morgan's law: Given two sets A, B,

$$(A \cap B)^c = A^c \cup B^c.$$
$$(A \cup B)^c = A^c \cap B^c.$$

<u>Homework</u>: Given $A_1, \cdots, A_n, \cdots$,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c.$$
$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

(6) Functions between sets:

Given two sets A, B, a <u>function</u> $f: A \to B$ is rule or mapping that takes each $x \in A$ to a single element in B, usually denoted $f(x)$.

A — domain of f

B — codomain of f.

$\{y \in B : y = f(x) \text{ for some } x\}$ — range of f.

<u>Example</u>: Dirichlet function:

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Domain: $\mathbb{R}$. Codomain: $\mathbb{R}$ Range: $\{0, 1\}$.
by default.

Rmk: ① No closed formula, like $\sqrt{1+x^2}$, exists for $D(x)$.

② An example that you should always keep in mind.

Example: Absolute Value:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Domain: $\mathbb{R}$. Codomain: $\mathbb{R}$ by default. Range: $\mathbb{R}_+$.

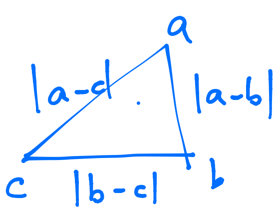
Properties: (1) $|ab| = |a| \cdot |b|$.

(2) $|a+b| \leq |a| + |b|$.

Most used form: $|a-b| \leq |a-c| + |c-b|$.

$LHS = |a-c+c-b| \leq |a-c| + |c-b| = RHS$

Why is it called triangle inequality?



Just pretend a, b, c are points in the plane, $|a-b|$, $|b-c|$, $|a-c|$ are distances between these points.

Homework: 1, 5, 6, 7.